Unitarity in Scalar-Metric Quantum Cosmology

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Outline of the presentation

- Brief introduction on canonical quantum gravity and quantum cosmology.
- We shall pick up EH gravity theory along with a scalar field and cosmic matter part present in it.
- We also consider an anisotropic cosmological model.
- We shall describe Schutz's formalism for the matter sector.
- Wheeler-DeWitt quantization.
- Restoring the unitarity.

- Gravity is the geometry of curved space-time.
- Mass-energy curves the space-time.
- Free mass moves on straight paths on curved space-time.

Quantum theory of gravity essentially means quantum theory of space-time geometry.

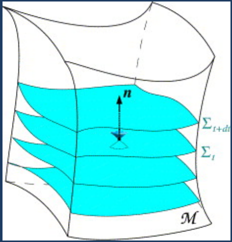
- Is this possible to test quantum gravity in laboratory?

$$E_{pl} \equiv \sqrt{\hbar C^5/G} \approx 10^{19} GeV$$

- Cosmological laboratory.
- Removing the Big-Bang singularity.
- Understanding the birth of the universe.

- The starting point of quantum gravity is the Hamiltonian formalism of gravity.
- ADM formalism : Space and time separation ($3+1$ split).
- ADM decomposition of the metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_a N^a & N^b \\ N_b & h_{ab} \end{pmatrix}$$



where N is lapse function, N_a is shift vector and h_{ab} is induced metric on the 3-d hypersurface foliated at fixed time.

- Superspace : The space of 3-geometries. (J. A. Wheeler & B. S. DeWitt)
- Quantum cosmology is performed on finite dimensional mini-superspace.

Let us start with the action

$$\mathcal{A} = \int_M d^4x\sqrt{-g}\Big[R - F(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] + 2\int_{\partial M} d^3x\sqrt{h}h_{ij}K^{ij} + \int_M d^4x\sqrt{-g}P \quad (1)$$

- R is the ricci scalar with the metric $g_{\mu\nu}$.
- Second term is a non-linear self-coupling scalar field minimally coupled to gravity.
- The second integration term is the GHY term where h_{ij} and K are the induced metric and extrinsic curvature of the fixed time sliced hypersurface respectively.
- P is the pressure of the cosmological fluid.

Reference - Babak Vakili, Phys. Lett. B 688 (2010) 129.
J. Socorro, M. Sabido, M.A. Sanchez, M.G. Frias Palos, Rev. Mex. Fs. 56 (2) (2010) 166171.

- Here we shall consider the cosmic matter as a perfect fluid obeying EoS

$$P = \omega\rho \; . \tag{2}$$

ρ being the density of the fluid.

- In Schutz's formalism one can cast fluid's four velocity vector in terms of four potentials h, ϵ, θ and S in the following way

$$u_\nu = \frac{1}{h}(\partial_\nu\epsilon + \theta\partial_\nu S) \tag{3}$$

where h is the specific enthalpy and S is the specific entropy. The other two potentials θ and ϵ are irrelevant physically.

- Normalozation condition reads

$$u_\nu u^\nu = 1 \; . \tag{4}$$

Reference - B. F. Schutz, Phys. Rev. D 2 (1970) 2767.

- Now the cosmic fluid pressure P in terms of the specific enthalpy h and specific entropy S reads

$$P = \frac{\omega}{(1+\omega)^{1+1/\omega}} h^{1+1/\omega} e^{-S/\omega} \ . \tag{5}$$

- <u>Cosmological model</u> : Here we take up Bianchi I metric which is given by

$$ds^2 = N^2(t)dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2 \tag{6}$$

where $N(t)$ is called the lapse function and $A(t), B(t), C(t)$ are three functions of the cosmic time t.

Reference - B. F. Schutz, Phys. Rev. D 4 (1971) 3559.

- The Ricci scalar for this metric is given by

$$R = \frac{-2}{N^3ABC}\left[NA\dot{B}\dot{C} + B\left(N\dot{A}\dot{C} + NA\ddot{C} - \dot{N}A\dot{C}\right) + C\left\{N\left(B\ddot{A} + \dot{A}\dot{B} + A\ddot{B}\right) - \dot{N}\left(B\dot{A} + A\dot{B}\right)\right\}\right] \tag{7}$$

where the dots denote derivative with respect to time t.

- The gravity sector of the action along with the scalar field can be written down upto a constant volume factor as

$$\begin{aligned} S_g &= \int dt\left[-\frac{2}{N}(\dot{A}\dot{B}C + \dot{B}\dot{C}A + \dot{C}\dot{A}B) - \frac{F(\phi)ABC}{N}\dot{\phi}^2\right] \\ &= \int dt\ L_g\ . \end{aligned} \tag{8}$$

- Once the Lagrangian for the gravity part is identified, we can proceed to find out the Hamiltonian for the gravity sector.

- We make the following transformations

$$A(t) = e^{Z_0+Z_++\sqrt{3}Z_-}$$
$$B(t) = e^{Z_0+Z_+-\sqrt{3}Z_-}$$
$$C(t) = e^{Z_0-2Z_+} \tag{9}$$

where $Z_0(t), Z_+(t), Z_-(t)$ are the new variables that we shall work with instead of $A(t), B(t), C(t)$.

- The Hamiltonian for the gravity sector therefore reads

$$H_g = -\frac{1}{24}Ne^{-3Z_0}(p_0^2 - p_+^2 - p_-^2) - \frac{1}{4F(\phi)}Ne^{-3Z_0}p_\phi^2 \tag{10}$$

where p_0, p_+, p_- and p_ϕ are the canonical momenta conjugate to Z_0, Z_+, Z_- and ϕ respectively.

Reference - F. G. Alvarenga, J. C. Fabris, N. A. Lemos, G. A. Monerat, Gen. Relativ. Gravit. 34 (2002) 651.

- With respect to a comoving observer, the fluid four velocity vector takes the form $u_\nu = (N, 0, 0, 0)$. Using eq.(s)(3), (4), we obtain

$$h = \frac{\dot{\epsilon} + \theta \dot{S}}{N} \ . \tag{11}$$

- Substituting h in eq.(5) leads to the form of the matter sector of the action (1) which upto a volume factor reads

$$\begin{aligned} S_m &= \int dt \left[N(t)^{-1/\omega} e^{3Z_0} \frac{\omega}{(1+\omega)^{1+1/\omega}} (\dot{\epsilon} + \theta \dot{S})^{1+1/\omega} e^{-S/\omega} \right] \\ &= \int dt \ L_m \ . \end{aligned} \tag{12}$$

- The Hamiltonian for the matter sector can be obtained as

$$H_m = N e^{-\omega Z_0} p_\epsilon^{\omega+1} e^S \tag{13}$$

- One can recast the Hamiltonian for the matter sector in a more tractable form. For that one needs the canonical transformations

$$\begin{aligned} T &= p_S e^{-S} p_\epsilon^{-(\omega+1)} && (14a) \\ p_T &= p_\epsilon^{\omega+1} e^S && (14b) \\ \bar{\epsilon} &= \epsilon - (\omega+1)\frac{p_S}{p_\epsilon} && (14c) \\ \bar{p_\epsilon} &= p_\epsilon \; . && (14d) \end{aligned}$$

- The Hamiltonian for the matter sector now becomes

$$H_m = N e^{-3Z_0} e^{3(1-\omega)Z_0} p_T \qquad (15)$$

where p_T is the canonical momentum conjugate to the variable T which can be considered as the new cosmic time.

Reference - V. G. Lapchinskii, V. A. Rubakov, Theor. Math. Phys. 33 (1977) 1076.

- The Hamiltonian for the full theory takes the form

$$\begin{aligned} H &\equiv H_g + H_m \\ &= Ne^{-3Z_0}\left[-\frac{1}{24}(p_0^2 - p_+^2 - p_-^2) - \frac{1}{4F(\phi)}p_\phi^2 + e^{3(1-\omega)Z_0}p_T\right] . \end{aligned} \tag{16}$$

- The gauge choice $N = e^{3\omega Z_0}$ makes the new canonical variables (T, p_T) decouple from the gravity sector. So the new set of spacetime coordinates are (Z_0, Z_+, Z_-, T).
- Quantization of the model : To get the WD equation, we first replace the momenta appearing in the Hamiltonian (16) by their quantum mechanical operator representations, namely,
$p_0 = -i\frac{\partial}{\partial Z_0}$, $p_+ = -i\frac{\partial}{\partial Z_+}$, $p_- = -i\frac{\partial}{\partial Z_-}$, $p_\phi = -i\frac{\partial}{\partial \phi}$ and $p_T = -i\frac{\partial}{\partial T}$ respectively (setting $\hbar = 1$).

- The WD equation then reads

$$\hat{H}\Psi(Z_0, Z_+, Z_-, T) = 0 \tag{17}$$

where

$$\hat{H} = \left[\frac{\partial^2}{\partial Z_0^2} - \frac{\partial^2}{\partial Z_+^2} - \frac{\partial^2}{\partial Z_-^2} + \frac{1}{4F(\phi)}\frac{\partial^2}{\partial \phi^2} - 24ie^{3(1-\omega)Z_0}\frac{\partial}{\partial T}\right] . \tag{18}$$

- We shall now consider a stiff fluid for which $\omega = 1$. The WD equation then reduces to

$$\frac{\partial^2 \Psi}{\partial Z_0^2} - \frac{\partial^2 \Psi}{\partial Z_+^2} - \frac{\partial^2 \Psi}{\partial Z_-^2} + \frac{1}{4F(\phi)}\frac{\partial^2 \Psi}{\partial \phi^2} = 24i\frac{\partial \Psi}{\partial T} . \tag{19}$$

- We now make the following ansatz to solve (19)

$$\Psi(Z, \phi, T) = e^{-iET}\Phi(Z, \phi) \ , Z \equiv (Z_0, Z_+, Z_-) \ . \tag{20}$$

- This yields

$$\hat{\mathcal{H}}\Phi = 24E\Phi \tag{21}$$

where

$$\hat{\mathcal{H}} = \frac{\partial^2}{\partial Z_0^2} - \frac{\partial^2}{\partial Z_+^2} - \frac{\partial^2}{\partial Z_-^2} + \frac{1}{4F(\phi)}\frac{\partial^2}{\partial \phi^2} \ . \tag{22}$$

- <u>Hermiticity</u> : To construct a well behaved wave function the operator $\hat{\mathcal{H}}$ has to be a self-adjoint operator. That is we must have

$$(\hat{\mathcal{H}}\Phi_1, \Phi_2) = (\Phi_1, \hat{\mathcal{H}}\Phi_2) \ . \tag{23}$$

- We define the inner product between any two wave functions Φ_1 and Φ_2 in the following way

$$(\Phi_1, \Phi_2) = \int \Phi_1^*(Z,\phi)F(\phi)\Phi_2(Z,\phi)\ dZd\phi \ . \tag{24}$$

- <u>BOUNDARY CONDITIONS</u> : $\Phi_2 = 0$, $\frac{\partial \Phi_2}{\partial Z_0} = 0$ at $Z_0 = \pm\infty$. The conditions are same for Z_+,Z_- and ϕ except for ϕ the end points are 0 and ∞.
- We apply the separation of variables and the partial differential equation (21) decouples to the following second order differential equations

$$\frac{d^2\eta(\phi)}{d\phi^2} + 4\kappa^2 F(\phi)\eta(\phi) = 0 \tag{25a}$$

$$\frac{d^2\xi_+(Z_+)}{dZ_+^2} + K_+^2\xi_+(Z_+) = 0 \tag{25b}$$

$$\frac{d^2\xi_-(Z_-)}{dZ_-^2} + K_-^2\xi_-(Z_-) = 0 \tag{25c}$$

$$\frac{d^2\xi_0(Z_0)}{dZ_0^2} + (K_+^2 + K_-^2 - \kappa^2 - 24E)\xi_0(Z_0) = 0\ . \tag{25d}$$

- Assume $F(\phi)=\frac{\lambda}{4}\phi^m, (m\neq -2, \lambda>0)$ along with the boundary conditions the solutions of (25) lead to the total wave function of the form

$$\Psi(Z,\phi,T)=C_0C_+C_-C_{m,\lambda}\kappa^{\frac{1}{m+2}}\phi^{\frac{1}{2}}e^{-iK_+Z_+}e^{-iK_-Z_-}e^{-iK_0Z_0} \times e^{-iET}J_{\frac{1}{m+2}}\left(\frac{2\sqrt{\lambda}\phi^{\frac{m+2}{2}}\kappa}{m+2}\right) \tag{26}$$

where

$$C_{m,\lambda}=c_2(m+2)^{-\frac{1}{m+2}}\lambda^{\frac{1}{2(m+2)}}\Gamma\left(1+\frac{1}{m+2}\right).$$

- We now proceed to construct a wave packet using the superposition principle in the following way

$$\Psi_{wp}=\int\kappa^{\frac{1}{2}-\gamma}e^{-(K_0^2+K_+^2+K_-^2+\kappa^2)}\Psi(Z,\phi,T)\,d\kappa dK_0dK_+dK_-\,. \tag{27}$$

- Important note :

$$K_0^2 = K_+^2 + K_-^2 - \kappa^2 - 24E \tag{28}$$

- With this wave packet, we calculate its norm. This reads

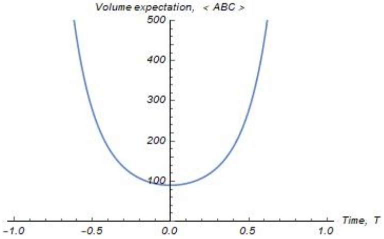
$$||\Psi_{wp}|| = \frac{1}{8}\left(\frac{C_0 C_+ C_- C_{m,\lambda}}{2\gamma}\right)^2 \left(\sqrt{\frac{\pi}{2}}\right)^4 . \tag{29}$$

- So now the normalized wave packet becomes

$$\Psi_{wp} = \frac{8\sqrt{2}\gamma}{\pi C_0 C_+ C_- C_{m,\lambda}} \int \kappa^{\frac{1}{2}-\gamma} e^{-(K_0^2+K_+^2+K_-^2+\kappa^2)} \Psi(Z,\phi,T) \, d\kappa dK_0 dK_+ dK_- \, . \tag{30}$$

- From this wave packet (30), one can as well proceed to calculate the expectation value of the spatial volume of the universe. This reads

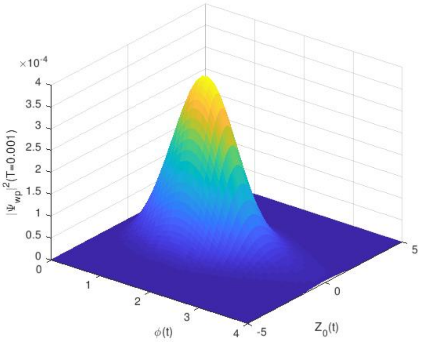
$$\langle ABC\rangle(T) \equiv \langle e^{3Z_0}\rangle(T) = e^{\frac{9}{2}\left(T^2+1\right)}. \quad (31)$$



- It clearly tells us that at the beginning of time, that is at $T = 0$, the universe had a finite volume. The figure displays the variation of the volume expectation of the universe with the time parameter T.

- Here now let's study the behavior of the probability density function, that is

$$\rho = \Psi^*_{wp}\Psi_{wp} \tag{32}$$



Behavior of the probability density function with respect to ϕ and Z_0. We plot for a particular value of the time parameter $T = 0$, with the other constant values $\lambda = 1$ and $m = 2$.

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Thank You